Failure avalanches in fiber bundles for discrete load increase

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(Received 10 January 2007; published 2 April 2007)

The statistics of burst avalanche sizes n during failure processes in a fiber bundle follows a power law $D(n)\sim n^{-\xi}$ for large avalanches. The exponent ξ depends upon how the avalanches are provoked. While it is known that when the load on the bundle is increased in a continuous manner the exponent takes the value $\xi=5/2$, we show that when the external load is increased in discrete and not too small steps, the exponent value $\xi=3$ is relevant. Our analytic treatment applies to bundles with a general probability distribution of the breakdown thresholds for the individual fibers. The preasymptotic size distribution of avalanches is also considered.

DOI: 10.1103/PhysRevE.75.046101 PACS number(s): 62.20.Mk

I. INTRODUCTION

Fiber bundles with statistically distributed thresholds for the breakdown of individual fibers provide simple and interesting models of failure processes in materials under stress. These models are much studied since they can be analyzed to an extent that is not possible for more complex materials (for reviews see [1–5]). We study here equal-load-sharing models, in which the load previously carried by a failed fiber is shared equally by all remaining intact fibers in the bundle [6–9]. The breakdown thresholds x_i for the individual fibers are assumed to be independent random variables with the same cumulative distribution function $P(x)$ and a corresponding probability density $p(x)$:

$$\mathrm{Prob}(x_i \leqslant x) = P(x) = \int_0^x p(y)dy. \tag{1}$$

We consider a bundle of N fibers, clamped at both ends. At a force x per surviving fiber the total force $F(x)$ on the bundle is x times the number of intact fibers. The average, or macroscopic, force is given by the expectation value of this,

$$\langle F\rangle = Nx[1-P(x)]. \tag{2}$$

One may consider x to represent the elongation of the bundle, with the elasticity constant set equal to unity. In the generic case $\langle F\rangle$ will have a single maximum F_c, a critical load corresponding to the maximum load the bundle can sustain before complete breakdown of the whole system.

When a fiber ruptures somewhere, the stress on the intact fibers increases. This may in turn trigger further fiber failures, which can produce avalanches that lead either to a stable situation or to a breakdown of the whole bundle. One may study the burst distribution $D(n)$, defined as the expected number of bursts of size n when the bundle is stretched until complete breakdown. When the load on the bundle is increased continuously from zero, the generic result is a power law

$$\lim_{N\to\infty} N^{-1}D(n) \propto n^{-\xi}, \tag{3}$$

for large n, with $\xi=5/2$ [10,11].

However, experiments may be performed in a different manner. In Sec. II we show for the uniform probability distribution of thresholds that for experiments in which the load is increased in finite steps of size δ rather than by infinitesimal amounts, the power-law exponent is seen to increase to the value $\xi=3$. This has been noticed in a special case previously: An argument by Pradhan *et al.* [12] for uniform threshold distributions suggested this exponent value.

In Sec. III we show that the same asymptotic power law

$$D(n) \propto n^{-3} \tag{4}$$

holds for a triangular threshold distribution and for a Weibull distribution. The result (4) is demonstrated by simulations, and we provide analytic derivations to back up the results. Our conclusions are not limited to the distributions used in the simulations; we give an analytic derivation valid for a *general* threshold distribution.

The avalanche distribution does not follow the asymptotic power law (4) for small bursts. However, our analytic formulas do also cover the distribution for bursts of smaller sizes n, down to a minimum size.

In the concluding remarks we discuss briefly the results and their range of applicability.

II. UNIFORM THRESHOLD DISTRIBUTION

For the uniform distribution of thresholds,

$$P(x) = \begin{cases} x & \text{for } 0 \leqslant x \leqslant 1, \\ 0 & \text{for } x > 1, \end{cases} \tag{5}$$

here in dimensionless units, the load curve (2) is parabolic,

$$\langle F\rangle = Nx(1-x), \tag{6}$$

so that the expected critical load equals

$$F_c = N/4. \tag{7}$$

Figure 1 shows the size distribution of the bursts obtained by simulation on fiber bundles with the uniform threshold dis-

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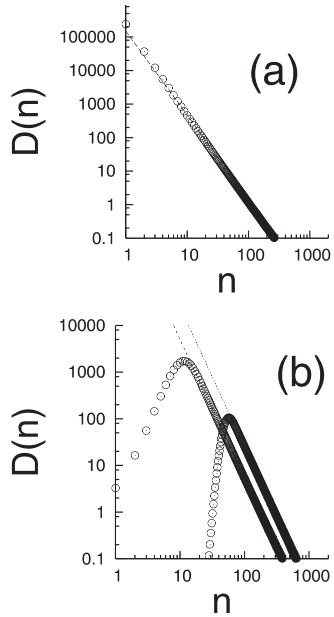

FIG. 1. Avalanche size distribution for the uniform threshold distribution (5). The distribution marked (a) is a record of all bursts when the load is increased continuously, while the bursts recorded in (b) result from increasing the load in steps of $\delta=10$ and $\delta=50$ (upper curve). The dotted line in (a) represents a $n^{-5/2}$ behavior, and the dotted lines in (b) show the theoretical asymptotics (12) for $\delta=10$ and $\delta=50$. The figure is based on 10 000 samples with $N=10^6$ fibers in the bundle.

tribution (5). In one procedure the bursts are recorded under a continuous load increase; in the other procedure the load is increased by discrete amounts δ. Clearly the exponent describing the large-size power laws are different, close to $\xi=2.5$ and $\xi=3$, respectively.

The basic reason for the difference in the power laws is that increasing the external load in steps reduces the fluctuations in the force. The derivation of the asymptotic size distribution $D(n)\propto n^{-5/2}$ of avalanches, corresponding to load increases by infinitesimal steps, shows the importance of force fluctuations [10]. An effective reduction of the fluctuations requires that the size δ of the load increase be large enough so that a considerable number of fibers break in each step—i.e., $\delta \gg F_c/N$.

With a sufficiently large δ we may use the macroscopic load equation (2) to determine the number of fibers broken in each step. The load values are $m\delta$, with m taking the values $m=0,1,2,\ldots,N/4\delta$ for the uniform threshold distribution. By Eq. (6) the threshold value corresponding to the load $m\delta$ is

$$x_m = \frac{1}{2}(1-\sqrt{1-4m\delta/N}). \tag{8}$$

The expected number of fibers broken when the load is increased from $m\delta$ to $(m+1)\delta$ is close to

$$n = N dx_m/dm = \delta/\sqrt{1-4m\delta/N}. \tag{9}$$

Here the minimum number of n is δ, obtained in the first load increase. The integral over all m from 0 to $N/4\delta$ yields a total number $N/2$ of broken fibers, as expected, since the remaining one-half of the fibers burst in one final avalanche.

The number of avalanches of size between n and $n+dn$, $D(n)\,dn$, is given by the corresponding interval of the counting variable m:

$$D(n)dn = dm. \tag{10}$$

Since

$$\frac{dn}{dm} = \frac{2\delta^2}{N}(1-4m\delta/N)^{-3/2} = \frac{2}{N\delta}n^3, \tag{11}$$

we obtain the following distribution of avalanche sizes:

$$D(n) = \frac{dm}{dn} = \frac{1}{2}N\delta n^{-3} \quad (n \geqslant \delta). \tag{12}$$

For consistency, one may estimate the total number of bursts by integrating $D(n)$ from $n=\delta$ to ∞, with the result $N/4\delta$, as expected.

Figure 1 shows that the theoretical power law (12) fits the simulation results perfectly for sufficiently large n. The simulation records also a few bursts of magnitude less than δ because there is a nonzero probability to have bundles with considerably fewer fibers than the average in a threshold interval. However, these events will be of no importance for the asymptotic power law in the size distribution. Moreover, for a more realistic threshold distribution the situation is different (see below).

III. GENERAL THRESHOLD DISTRIBUTIONS

In order to see whether the asymptotic law (4) is general, we have performed simulations for two other threshold distributions: the triangular threshold distribution

$$p(x) = \begin{cases} 2x & \text{for } 0 \leqslant x \leqslant 1, \\ 0 & \text{for } x > 1, \end{cases} \tag{13}$$

and the Weibull distribution with index 5,

$$P(x) = 1 - e^{-x^5}, \quad \text{for all positive } x. \tag{14}$$

In Fig. 2 the exponent $\xi=3$ is clearly present in the triangular distribution case and less clearly in the Weibull distribution case. Therefore we give now an analytic derivation for a general threshold distribution and apply the result to the two cases for which the simulation results are shown in Fig. 2. The intention is to see if the simulation results can be fitted, not merely with a power-law behavior for large avalanche sizes, but over an extended range of n.

For a general threshold distribution $P(x)$ a load interval δ and a threshold interval are connected via the load equation (2). Since $d\langle F\rangle/dx = N[1-P(x)-xp(x)]$, an increase δ in the load corresponds to an interval

$$\Delta x = \frac{\delta}{N[1-P(x)-xp(x)]} \tag{15}$$

of fiber thresholds. The expected number of fibers broken by this load increase is therefore

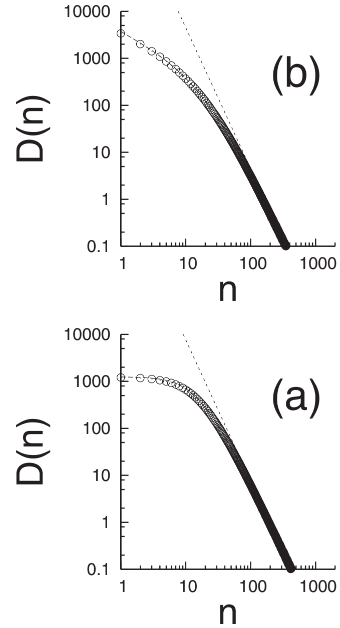

FIG. 2. Avalanche size distribution for (a) the triangular distribution (13) and (b) for the Weibull distribution (14). Open circles represent simulation data, dashed lines are analytic expressions [Eq. (22) in (a) and Eq. (23) in (b)], and dotted lines are the asymptotic power laws with exponent −3. In both cases the load is increased in steps of $\delta=20$. The figure is based on 10 000 samples of bundles with $N=10^6$ fibers.

$$n = Np(x)\Delta x = \frac{p(x)}{1-P(x)-xp(x)}\delta. \tag{16}$$

Note that this number diverges at the critical point—i.e., at the maximum of the load curve—as expected.

We want to determine the number $D(n)\,\Delta n$ of bursts with magnitudes in the interval $(n, n+\Delta n)$. The number $D(n)\,\Delta n$ we seek is the number of load steps corresponding to the interval Δn. Since each load step corresponds to the interval (15) of x, the number of load steps corresponding to the interval Δn equals

$$D(n)\Delta n = \Delta n \frac{dx}{dn}\frac{N[1-P(x)-xp(x)]}{\delta}. \tag{17}$$

Using Eq. (16) we have

$$\frac{dn}{dx} = \frac{[1-P(x)]p'(x)+2p(x)^2}{[1-P(x)-xp(x)]^2}\delta. \tag{18}$$

Thus we obtain

$$D(n) = \frac{[1-P(x)-xp(x)]^3}{[1-P(x)]p'(x)+2p(x)^2}\frac{N}{\delta^2} = \frac{p(x)^3}{[1-P(x)]p'(x)+2p(x)^2}\frac{N\delta}{n^3}, \tag{19}$$

using Eq. (16). Here $x=x(n)$, determined by Eq. (16). For the uniform threshold distribution expression (19) coincides with Eq. (12).

Near criticality the first fraction in the expression (19) for $D(n)$ becomes a constant. The asymptotic behavior for large n is therefore

$$D(n) \simeq Cn^{-3}, \tag{20}$$

with a nonzero constant

$$C = N\delta \frac{p(x_c)^2}{2p(x_c)+x_c p'(x_c)}. \tag{21}$$

We have used that at criticality $1-P(x_c)=x_c p(x_c)$. Thus the asymptotic power law (4) is universal.

For the triangular threshold distribution considered in Fig. 2(a), Eqs. (16) and (19) yield $n=2x\delta/(1-3x^2)$ and $D=4N\delta n^{-3}x^3/(1+3x^2)$. Elimination of x gives

$$D(n) = \frac{2N\delta}{n^3}\frac{1}{6\delta/n+2\delta^3/n^3+(3+2\delta^2/n^2)\sqrt{3+\delta^2/n^2}}, \tag{22}$$

with asymptotic behavior $D(n)\simeq(2N\delta/\sqrt{27})n^{-3}$, in agreement with (20).

For the Weibull distribution considered in Fig. 2(b) we obtain

$$D(n) = N\delta n^{-3}\frac{25x^9e^{-x^5}}{4+5x^5}, \quad n = \frac{5\delta x^4}{1-5x^5}. \tag{23}$$

This burst distribution must be given in parameter form; the elimination of x cannot be done explicitly. The critical point is at $x=5^{-1/5}$ and the asymptotics is given by (20), with $C=N\delta(625e)^{-1/5}$.

These theoretical results are also exhibited in Fig. 2. In both cases the agreement with the theoretical results is very satisfactory. The n^{-3} power law is seen, but the asymptotics sets in only for very large avalanches, especially for the Weibull case. For small-sized avalanches it is interesting to note the difference between the uniform distribution and the more realistic Weibull distribution. In the latter case the theoretical result (19) is reasonably accurate also for $n<\delta$. The reason is that for the Weibull (and the triangular) threshold distribution there are few very weak fibers. Thus a load increase δ corresponds to a threshold interval that may, in the average description, contain just a few failing fibers. For the uniform distribution, on the other hand, a corresponding threshold interval with a number of fibers much less than δ can only be caused by fluctuations with very low probabilities.

IV. CONCLUDING REMARKS

If we let the load increase δ shrink to zero, we must recover the asymptotic $D(n)\propto n^{-5/2}$ power law valid for continuous load increase. Thus, as function of δ, there must be a crossover from one behavior to the other. It is to be expected that for $\delta \ll 1$ the $D(n)\propto n^{-5/2}$ asymptotics is seen, and when $\delta \gg 1$ the $D(n)\propto n^{-3}$ asymptotics is seen.

We would like to emphasize that the stepwise load increase is a reasonable, as well as more practical, loading method from the experimental point of view. While performing fracture-failure experiments by applying an external load, one cannot ensure that a single fiber (the weakest one) fails, whereas increasing the external load by equal steps is an easier and more realistic procedure.

In conclusion, we have shown that the magnitude distribution $D(n)$ of avalanches generated by simulation for fiber bundles with stepwise increase in the external load is asymptotically a power law with an exponent essentially equal to -3. We have analytically derived this asymptotic power law $D(n)=Cn^{-3}$ for a general probability distribution $p(x)$ of the individual fiber thresholds, as well as the preasymptotic behavior of the avalanche distribution $D(n)$.

ACKNOWLEDGMENTS

We would like to thank A. Hansen for useful comments. S.P. thanks the Research Council of Norway (NFR) for financial support through Grants Nos. 166720/V30 and 177958/V30.

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